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Generic automorphisms and graph coloring

James H. Schmerl

Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009, USA

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Abstract

The question of whether or not a given countable arithmetically saturated model of Peano Arithmetic has a generic automorphism is shown to be very closely connected to Hedetniemi's well-known conjecture on the chromatic number of products of graphs.

Keywords: Generic automorphisms; Hedetniemi's conjecture; Recursively saturated; Peano Arithmetic

Generic automorphisms are very useful in the study of the automorphism groups of countable first-order structures. The book [6] contains much of interest concerning these groups. A good source of structures having rich and interesting automorphism groups are the $\aleph_0$-categorical ones, about which much information can be found by Evans [3] and Cameron [1]. Two examples we mention are the rationals $(\mathbb{Q}, <)$, considered as a linearly ordered set, and the random graph. Both of these have generic automorphisms. Another good source of structures having rich and interesting automorphism groups are the recursively saturated ones, which are also discussed in [6], especially models of Peano Arithmetic. Automorphism groups of countable recursively saturated of PA have been given a great deal of consideration, and it is the automorphisms of such structures which will be dealt with here.

Throughout this paper, $\mathscr{M}$ will be a countable, recursively saturated model of Peano Arithmetic. We consider $\mathrm{Aut}(\mathscr{M})$, the topological group of automorphisms of $\mathscr{M}$ whose basic open subgroups are the stabilizers of finite sets. Following Truss [12], we say the

E-mail address: schmerl@math.uconn.edu (J.H. Schmerl).

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automorphism $g \in \mathrm{Aut}(\mathscr{M})$ is *generic* if the set $\{f^{-1}gf : f \in \mathrm{Aut}(\mathscr{M})\}$ of its conjugates is a comeager subset of $\mathrm{Aut}(\mathscr{M})$.

We are interested in the following question.

Question A. *Which $\mathscr{M}$ have generic automorphisms*?

Lascar [9] has defined a different notion of generic. To distinguish Lascar's generic automorphisms from Truss's, we will refer to automorphisms which are generic in the sense of Lascar as being *Lascar-generic*. Lascar-generics are very important in his proof that all arithmetically saturated $\mathscr{M}$ have the small index property. Lascar's result, in turn, was needed in the proof in [7] that there is diversity in the class of automorphism groups of arithmetically saturated $\mathscr{M}$. The set of Lascar-generics is comeager and closed under conjugation. Thus, it is evident that for arithmetically saturated $\mathscr{M}$, if there are generics, then they are precisely the Lascar-generics. However, it is not at all evident that, in general, generics exist, even if there are Lascar-generics. The following proposition gives a condition which implies the existence of generics.

Proposition 1. *If $\mathscr{M}$ is arithmetically saturated and has an automorphism whose set of conjugates is dense, then every Lascar-generic automorphism is generic.*

Corollary 2. *If $\mathscr{M}$ is arithmetically saturated and has an automorphism whose set of conjugates is dense, then $\mathscr{M}$ has a generic automorphism.*

The proof of Proposition 1 follows quite easily from [9]. We give a proof, assuming the reader's familiarity with [9].

Proof of Proposition 1. Suppose that $\mathscr{M}$ is arithmetically saturated and that h is an automorphism whose set of conjugates is dense. Let $g_1, g_2 \in \mathrm{Aut}(\mathscr{M})$ be Lascar-generics, with the intent of showing that g_1, g_2 are conjugates. Then there are small $\mathscr{M}_1, \mathscr{M}_2 \prec \mathscr{M}$ such that $g_1|M_1$ and $g_2|M_2$ are existentially closed. Let $a_1, a_2 \in M$ be such that $M_1 = \{(a_1)_i : i \in \omega\}$ and $M_2 = \{(a_2)_i : i \in \omega\}$. Let h_1, h_2 be conjugates of h such that $h_1(a_1) = g_1(a_1)$ and $h_2(a_2) = g_2(a_2)$, and let f be such that $h_2 = f^{-1}h_1 f$. Then $h_1(f(a_2)) = f(g_2(a_2))$. As the set of Lascar-generics is dense, there is a Lascar-generic g agreeing with g_1 on $\{a_1, f(a_2)\}$, so that $g(a_1) = g_1(a_1)$ and $g(f(a_2)) = f(g_2(a_2))$. Thus, $f^{-1}gf(a_2) = g_2(a_2)$. Now, both g and $f^{-1}gf$ are Lascar-generic, and they agree with g_1 and g_2 on M_1 and M_2, respectively. Therefore, g and g_1 are conjugates and also $f^{-1}gf$ and g_2 are conjugates. Thus, g_1 and g_2 are conjugates. □

The previous corollary gives interest to the following question, which is a weakening of Question A since the set of conjugates of a generic automorphism is dense.

Question B. *For which $\mathscr{M}$ is there an automorphism whose set of conjugates is a dense subset of $\mathrm{Aut}(\mathscr{M})$*?

In the case that $\mathscr{M}$ is a model of True Arithmetic, there is the following definitive answer to Question B.

Theorem 3. *If $\mathscr{M} \models \mathsf{TA}$, then $\mathscr{M}$ has an automorphism whose set of conjugates is a dense subset of* $Aut(\mathscr{M})$.

This theorem yields the following corollary, giving a very partial answer to Question A.

Corollary 4. *If $\mathscr{M} \models \mathsf{TA}$ is arithmetically saturated, then $\mathscr{M}$ has a generic automorphism.*

In all cases not covered by Theorem 3, Question B is still open and is very closely tied to Hedetniemi's Conjecture, which is a well known and still open conjecture in the chromatic theory of graphs formulated more than 30 years ago. (See Section 11.1 of [5].)

A *digraph* D is a pair (V, E), where $E \subseteq V^2$ and $(x, x) \notin E$ whenever $x \in V$. The *direct product* $D_1 \times D_2$ of two digraphs $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ is the digraph $(V_1 \times V_2, E)$, where $E = \{((x_1, x_2), (y_1, y_2)) : (x_1, y_1) \in E_1$ and $(x_2, y_2) \in E_2\}$. A function $\alpha : V \longrightarrow C$ is a *C-coloring* of the digraph $D = (V, E)$ if $\alpha(x) \neq \alpha(y)$ whenever $(x, y) \in E$. The *chromatic number* $\chi(D)$ is the least cardinal κ for which there is a κ-coloring of D. It is very easy to see that $\chi(D_1 \times D_2) \leqslant \min(\chi(D_1), \chi(D_2))$.

For cardinals κ, λ, let $H(\kappa, \lambda)$ be the statement

Whenever D_1, D_2 are digraphs such that $\chi(D_1), \chi(D_2) \geqslant \kappa$, then $\chi(D_1 \times D_2) \geqslant \lambda$.

A crucial ingredient in the proof of Theorem 3 is the following proposition.

Proposition 5 (*Hajnal [4]*). $H(\aleph_0, \aleph_0)$.

It is to be noted that Proposition 5 is a consequence of a result about finite digraphs: If finite graphs D_1, D_2, where $D_1 = (V_1, E_1)$, are such that

$$\chi(D_2) > (\chi(D_1) - 1) \cdot \binom{|V_1|}{2},$$

then $\chi(D_1 \times D_2) \geqslant \chi(D_1)$.

If n, k are finite, then $H(n, k)$ is equivalent to: whenever D_1, D_2 are *finite* digraphs such that $\chi(D_1), \chi(D_2) \geqslant n$, then $\chi(D_1 \times D_2) \geqslant k$. Thus, the following proposition can be formalized in the language of PA and, in fact, is provable in PA.

Proposition 6. (1) *If $1 \leqslant k \leqslant 3$, then $H(k, k)$.*

(2) (Poljak [10]) *If $H(n + 1, 4)$, then $H(2^{2^n} + 1, 5)$.*

(3) (Poljak and Rödl [11]) *If $k > 4$ and $H(n + 1, k)$, then $H(2^n + 1, k + 1)$.*

The truth of the statement $\forall k < \omega \exists n < \omega H(n, k)$ (or of the statement $\exists n < \omega H(n, 4)$ to which it is equivalent by the previous proposition) is an open question. It is related to Hedetniemi's Conjecture and implies a weak, and still open, form of that conjecture.

We make a brief digression to discuss Hedetniemi's Conjecture. A *graph G* is a digraph (V, E) for which $(x, y) \in E$ iff $(y, x) \in E$. Let $H^*(\kappa, \lambda)$ be the statement:

Whenever G_1, G_2 are graphs such that $\chi(G_1), \chi(G_2) \geqslant \kappa$, then $\chi(G_1 \times G_2) \geqslant \lambda$.

It is easy to see that $H^*(n, n+1)$ is false for all $n < \omega$; in fact, for every n and every G_1, G_2, if $\chi(G_1), \chi(G_2) \leqslant n$ then $\chi(G_1 \times G_2) \leqslant n$. Hedetniemi's Conjecture is the statement: $\forall n < \omega H^*(n, n)$. It is obvious that $H^*(n, n)$ is true when $1 \leqslant n \leqslant 3$, and El-Zahar and Sauer [2] proved $H^*(4, 4)$. The conjecture is still open for all $n > 4$; in fact, even whether or not $\exists n H^*(n, 5)$ is unknown. A weak, and still open, form of Hedetniemi's Conjecture is the statement $\forall k < \omega \exists n < \omega H^*(n, k)$. It is clear that if $n, k < \omega$, then $H^*(n, k^2+1) \implies H(n, k+1) \implies H^*(n, k+1)$. Thus, the weak form of Hedetniemi's Conjecture is equivalent to $\exists n < \omega H^*(n, 10)$.

We can now state the theorem which gives some additional conditional answers to Question B.

Theorem 7. *If $\mathscr{M} \not\models \mathsf{TA}$, then the following are equivalent*:

(7.1) *$\mathscr{M}$ has an automorphism whose set of conjugates is dense.*
(7.2) *There is $n < \omega$ such that $\mathscr{M} \models H(n, 4)$.*

This theorem yields the following corollary, giving some additional conditional answers to Question A.

Corollary 8. *If $\mathscr{M} \not\models \mathsf{TA}$ is arithmetically saturated, then the following are equivalent*:

(8.1) *$\mathscr{M}$ has a generic automorphism.*
(8.2) *There is $n < \omega$ such that $\mathscr{M} \models H(n, 4)$.*

We now present proofs of Theorems 3 and 7. The following notation will be used: For elements a, b, c, d of $\mathscr{M}$, we will write $ab \equiv cd$ to mean that the types of $\langle a, b\rangle$ and $\langle c, d\rangle$ are the same.

Proof of Theorem 3. Let $\mathscr{M}$ be a countable, recursively saturated model of TA. We wish to build an automorphism f whose set of conjugates is dense. At some stage of the construction we have $a_1, a_2 \in M$ and have decreed that $f(a_1) = a_2$. We now consider some $b_1, b_2 \in M$ such that $b_1 \equiv b_2$. Our object is to find $b'_1, b'_2 \in M$ such that $b'_1 b'_2 \equiv b_1 b_2$ and $a_1 b'_1 \equiv a_2 b'_2$, and then we will decree, in addition, that $f(b'_1) = b'_2$.

Let $\Gamma(x_1, x_2, y_1, y_2)$ be the following set of formulas:

$$\{\varphi(x_1, x_2) : \mathscr{M} \models \varphi(a_1, a_2)\} \cup \{\psi(y_1, y_2) : \mathscr{M} \models \psi(b_1, b_2)\}$$
$$\cup \{\theta(x_1, y_1) \longleftrightarrow \theta(x_2, y_2) : \theta(x, y) \text{ a formula}\}.$$

If Γ is consistent, then we are done. For then Γ is realized in $\mathscr{M}$ by, for example, (c_1, c_2, d_1, d_2). Then $a_1 a_2 \equiv c_1 c_2$, so there is $g \in \mathrm{Aut}(\mathscr{M})$ such that $g(c_1) = a_1$ and $g(c_2) = a_2$. Then let $b'_1 = g(d_1)$ and $b'_2 = g(d_2)$.

We now show that Γ is consistent. Aiming for a contradiction, suppose that it is inconsistent. Then there are formulas

$$\varphi(x_1, x_2), \psi(y_1, y_2), \theta_0(x, y), \theta_1(x, y), \ldots, \theta_{m-1}(x, y)$$

such that $\mathscr{M}\models\varphi(a_1, a_2) \wedge \psi(b_1, b_2)$ and

$$\vdash\varphi(x_1, x_2) \wedge \psi(y_1, y_2) \longrightarrow \bigvee_{j<m} [\theta_j(x_1, y_1) \longleftrightarrow \neg\theta_j(x_2, y_2)].$$

For a first case, suppose that $\mathscr{M}\models\exists x\varphi(x, x) \vee \exists y\psi(y, y)$. Without loss of generality, suppose $\mathscr{M}\models\exists x\varphi(x, x)$, and let $c \in M$ be the least such that $\mathscr{M}\models\varphi(c, c)$. Then for some $j < m$, $\mathscr{M}\models\theta_j(c, b_1) \longleftrightarrow \neg\theta_j(c, b_2)$, contradicting that b_1, b_2 realize the same type.

Thus, for a second case, suppose that $\mathscr{M}\models\forall x\neg\varphi(x, x)\wedge\forall y\neg\psi(y, y)$. Define two digraphs as follows: $D_1=(M, E_1)$, where $(p_1, p_2) \in E_1$ iff $\mathscr{M}\models\varphi(p_1, p_2)$, and $D_2=(M, E_2)$, where $(q_1, q_2) \in E_2$ iff $\mathscr{M}\models\psi(q_1, q_2)$.

Then $\chi(D_1) = \chi(D_2) = \aleph_0$. For suppose not. Let $\chi(D_1) = n < \omega$. Then for each finite subdigraph D' of D_1, $\chi(D')\leqslant n$. Since $\mathscr{M}\models\mathsf{TA}$ and D_1 is 0-definable, it follows by overspill that

$$\mathscr{M}\models \text{“each finite subdigraph of } D_1 \text{ has an } n\text{-coloring”}$$

and, therefore, D_1 has an n-coloring $\alpha : M \longrightarrow n$ which is 0-definable in $\mathscr{M}$. Since $a_1 \equiv a_2$, it must be that $\alpha(a_1) = \alpha(a_2)$. However, $(a_1, a_2) \in E_1$, so this is a contradiction. Thus, $\chi(D_1) = \aleph_0$ and, similarly, $\chi(D_2) = \aleph_0$.

It follows from Proposition 5 that $\chi(D_1 \times D_2) = \aleph_0$. On the other hand, we can exhibit a 2^m-coloring β of $D_1 \times D_2$: For $(p, q) \in M \times M$, let $\beta(p, q)$ be the sequence $s : m \longrightarrow 2$, where $s_j = 1$ iff $\mathscr{M}\models\theta_j(p, q)$. Clearly, β is a coloring of $D_1 \times D_2$, yielding a contradiction and proving the theorem. $\square$

Proof of Theorem 7. We prove (7.1) $\Longrightarrow$ (7.2). Suppose that (7.2) is false. Then, by overspill, there is a nonstandard element c of $\mathscr{M}$ such that $\mathscr{M}\models\neg H(c, 4)$. Since $\mathscr{M}\not\models\mathsf{TA}$, we can find a definable nonstandard t such that $\mathscr{M}\models 2^t < c$. Thus $\mathscr{M}\models$“there are finite digraphs D_1, D_2 such that $\chi(D_1), \chi(D_2) > 2^t$ and $\chi(D_1 \times D_2)\leqslant 3$”. Then there are such digraphs $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ which are 0-definable. Let $\varphi_0(x)$ and $\psi_0(y)$ define V_1 and V_2, respectively, let $\rho_1(x_1, x_2)$ and $\rho_2(y_1, y_2)$ define E_1 and E_2, respectively, and let $\beta : V_1 \times V_2 \longrightarrow 4$ be a 0-definable 3-coloring of $D_1 \times D_2$.

Let $\theta_0(x), \theta_1(x), \theta_2(x), \ldots$ be a recursive list of all 1-ary formulas. We will recursively obtain sequences $\langle\varphi_i(x) : i < \omega\rangle$ and $\langle\psi_i(y) : i < \omega\rangle$ of 1-ary formulas such that for each $i < \omega$ the following hold:

(1) $\vdash\varphi_{i+1}(x) \longrightarrow \varphi_i(x)$ and $\vdash\psi_{i+1}(y) \longrightarrow \psi_i(y)$;
(2) either $\vdash\varphi_{i+1}(x) \longrightarrow \theta_i(x)$ or $\vdash\varphi_{i+1}(x) \longrightarrow \neg\theta_i(x)$;
(3) either $\vdash\psi_{i+1}(y) \longrightarrow \theta_i(y)$ or $\vdash\psi_{i+1}(y) \longrightarrow \neg\theta_i(y)$;
(4) $\mathscr{M}\models$“the subdigraphs of D_1 and D_2 induced by $\varphi_i(x)$ and $\psi_i(y)$, respectively, are not 2^{t-i}-colorable”.

Notice that we already have φ_0 and ψ_0 and that (4) holds when $i = 0$.

Now suppose that we have $\langle\varphi_i(x) : i \leqslant j\rangle$ and $\langle\psi_i(y) : i \leqslant j\rangle$, that (1)–(4) hold for $i < j$, and (4) holds for $i = j$. Since the graphs defined in (4) are not 2^{t-i}-colorable, it is possible to take φ_{j+1} to be either $\varphi_j(x) \wedge \theta_j(x)$ or $\varphi_j(x) \wedge \neg\theta_j(x)$ and to take $\psi_{j+1}(y)$ to be either $\psi_j(y) \wedge \theta_j(y)$ or $\psi_j(y) \wedge \neg\theta_j(y)$ so that (4) holds for $i = j + 1$.

Using the recursive saturation of $\mathscr{M}$ and the fact that each of the subdigraphs in (4) is not 1-colorable, we can find $a_1, a_2, b_1, b_2 \in M$ such that

$$\mathscr{M} \models \varphi_i(a_1) \wedge \varphi_i(a_2) \wedge \rho_1(a_1, a_2)$$

and

$$\mathscr{M} \models \psi_i(b_1) \wedge \psi_i(b_2) \wedge \rho_2(b_1, b_2)$$

for each $i < \omega$. From (2) we get that $a_1 \equiv a_2$ and from (3) that $b_1 \equiv b_2$.

We will show that there are no $a_1'a_2' \equiv a_1a_2$, $b_1'b_2' \equiv b_1b_2$ and automorphism *f* such that $f(a_1') = a_2'$ and $f(b_1') = b_2'$. For, given such a_1', a_2', b_1', b_2', we see that $((a_1', b_1'), (a_2', b_2')) \in E(D_1 \times D_2)$, so that $\beta(a_1', b_1') \neq \beta(a_2', b_2')$. But then $f(\beta(a_1', b_1')) = \beta(a_2', b_2') \neq \beta(a_1', b_1')$; therefore, *f* moves $\beta(a_1', b_1')$, contradicting that $\beta(a_1', b_1')$ is standard. Thus, there can be no such automorphism *f*.

Next we prove (7.2) $\Longrightarrow$ (7.1). We assume that $\mathscr{M} \not\models \mathsf{TA}$. Proceed as in the proof of Theorem 3, obtaining the set Γ, which we want to show to be consistent. Assuming that Γ is inconsistent, we proceed in the first case just as in the proof of Theorem 3. In the second case we get, as in the proof of Theorem 3, formulas

$$\varphi(x_1, x_2), \psi(y_1, y_2), \theta_0(x, y), \theta_1(x, y), \ldots, \theta_{m-1}(x, y)$$

and digraphs D_1 and D_2 such that

(1) for each $n < \omega$, $\mathscr{M} \models \chi(D_1) \geqslant n \wedge \chi(D_2) \geqslant n$;
(2) $\mathscr{M} \models \chi(D_1 \times D_2) \leqslant 2^m$.

As previously observed, such D_1 and D_2 can be found which are finite in the sense of $\mathscr{M}$. Thus, for no $n < \omega$ does $\mathscr{M} \models H(n, 2^m + 1)$ or, equivalently, for no $n < \omega$ does $\mathscr{M} \models H(n, 4)$. □

Lascar's proof in [9] used the existence of not just Lascar-generic automorphisms, but the existence of Lascar-generic *m*-tuples of automorphisms. For any $m \in \omega$, we can give $(\mathrm{Aut}(\mathscr{M}))^m$ the product topology. It then makes sense to refer to a dense set of *m*-tuples $\langle g_0, g_1, g_2, \ldots, g_{m-1}\rangle$ of automorphisms. We can also say that an *m*-tuple $\langle g_0, g_1, g_2, \ldots, g_{m-1}\rangle$ is *generic* if the set $\{\langle f^{-1}g_0f, f^{-1}g_1f, f^{-1}g_2f, \ldots, f^{-1}g_{m-1}f\rangle : f \in \mathrm{Aut}(\mathscr{M})\}$ of its conjugates is a comeager subset of $(\mathrm{Aut}(\mathscr{M}))^m$. All the previous results for automorphisms generalize to *m*-tuples of automorphisms.

We remark here that in general, a structure may have generic automorphisms but not generic *m*-tuples of automorphisms. For example, an unpublished result of Hodkinson, mentioned in [8], is that $(\mathbb{Q}, <)$ has no generic 2-tuples of automorphisms. On the other hand, the random graph does have generic *m*-tuples of automorphisms for each $m < \omega$.

There is no difficulty in straightforwardly generalizing Proposition 1 and Corollary 2 to *m*-tuples. For the other results, we will need to extend the notion of a digraph.

Suppose $2 \leqslant m \in \omega$. An *m-uniform hyperdigraph D* is a pair (V, E), where $E \subseteq V^m$ and $x_i \neq x_j$ whenever $\langle x_0, x_1, x_2, \ldots, x_{m-1}\rangle \in E$ and $i < j < m$. The *direct product* $D_1 \times D_2$ of two *m*-uniform hyperdigraphs $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ is the hyperdigraph $(V_1 \times V_2, E)$, where $E = \{\langle (x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_{m-1}, y_{m-1})\rangle :$

$\langle x_0, x_1, \dots, x_{m-1}\rangle \in E_1$ and $\langle y_0, y_1, \dots, y_{m-1}\rangle \in E_2\}$. A function $\alpha : V \longrightarrow C$ is a C-*coloring* of the hyperdigraph $D = (V, E)$ if whenever $\langle x_0, x_1, x_2, \dots, x_{m-1}\rangle \in E$, then it is not the case that $\alpha(x_0) = \alpha(x_1) = \alpha(x_2) = \cdots = \alpha(x_{m-1})$. Its *chromatic number* $\chi_m(D)$ is the least cardinal κ for which there is a κ-coloring of D.

Proposition 5 generalizes to n-uniform hyperdigraphs: *If D_1, D_2 are m-uniform hyperdigraphs such that $\chi_m(D_1), \chi_m(D_2) \geqslant \aleph_0$, then $\chi_m(D_1 \times D_2) \geqslant \aleph_0$*. This follows from the fact that if D_1, D_2 are finite m-uniform hyperdigraphs such that $D_1 = (V_1, E_1)$ and

$$\chi_m(D_2) > (\chi_m(D_1) - 1) \cdot \binom{|V_1|}{m},$$

then $\chi_m(D_1 \times D_2) \geqslant \chi_m(D_1)$. The following generalizations of Theorems 3 and Corollary 4 can then be proved.

Theorem 9. *If $\mathscr{M} \models \mathsf{TA}$, then there is an m-tuple of automorphisms of $\mathscr{M}$ whose set of conjugates is a dense subset of $(Aut(\mathscr{M}))^m$.*

Corollary 10. *If $\mathscr{M} \models \mathsf{TA}$ is arithmetically saturated, then there is a generic m-tuple of automorphisms of $\mathscr{M}$.*

Let $H_m(n, k)$ be the statement: If D_1, D_2 are m-uniform hyperdigraphs such that $\chi_m(D_1)$, $\chi_m(D_2) \geqslant n$, then $\chi_m(D_1 \times D_2) \geqslant k$. The truth of the sentences $\forall m < \omega \forall n < \omega H_m(n, n)$ and $\forall m < \omega \forall k < \omega \exists n < \omega H_m(n, k)$ seems to be unknown. In parallel with Theorem 7 and Corollary 8, we can get then the following results.

Theorem 11. *If $\mathscr{M} \not\models \mathsf{TA}$ and $1 \leqslant m < \omega$, then the following are equivalent*:

(11.1) *$\mathscr{M}$ has an m-tuple of automorphisms whose set of conjugates is a dense subset of $(Aut(\mathscr{M}))^m$.*
(11.2) *For each $k < \omega$ there is $n < \omega$ such that $\mathscr{M} \models H_m(n, k)$.*

Corollary 12. *If $\mathscr{M} \not\models \mathsf{TA}$ is arithmetically saturated and $1 \leqslant m < \omega$, then the following are equivalent*:

(12.1) *$\mathscr{M}$ has a generic m-tuple of automorphisms.*
(12.2) *For each $k < \omega$ there is $n < \omega$ such that $\mathscr{M} \models H_m(n, k)$.*

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